W^*-superrigidity for discrete quantum groups

Quantum Group Seminar

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Group von Neumann algebras

Let G be a countable group. The **left regular representation** is

$$\lambda : G \to \mathcal{U}(\ell^2(G)) : \lambda_g \delta_h = \delta_{gh} \, .$$

The **group von Neumann algebra** is

$$L(G) := \{\lambda_g \mid g \in G\}'' = \{u_g \mid g \in G\}'' = \overline{\mathrm{span}\{u_g \mid g \in G\}}^{\mathrm{SOT}} \, .$$

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Central Question

How much of G is remembered by $L(G)$?

Some background

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- $L(G_1 * G_2) \cong L(H_1 * H_2)$ for all infinite abelian groups G_1, G_2, H_1, H_2.

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- $L(G_1 * G_2) \cong L(H_1 * H_2)$ for all infinite abelian groups G_1, G_2, H_1, H_2.
- Open question: is $L(\mathbb{F}_2) \cong L(\mathbb{F}_3)$?

W*-superrigidity for groups

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Definition

A group G is called **W*-superrigid** if the following holds: whenever Λ is a groups s.t $L(G) \cong L(\Lambda)$, then $G \cong \Lambda$.

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Ioana, Popa, Vaes 2010: First examples of W*-superrigid groups.

Berbec, Vaes 2012: Simplified examples:

$$G := (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$$

is W*-superrigid whenever Γ is in 'class $\mathcal{C}$', e.g $\Gamma = \mathbb{F}_n$ for $2 \leq n \leq \infty$.

Overview of W*-superrigidity results

- *Ioana, Popa, Vaes 2010:* First examples (generalized wreath products).
- *Berbec, Vaes 2012:* Simplified examples (left-right wreath products).
- *Berbec 2014:* More left-right wreath product examples.
- *Chifan, Ioana 2017:* First amalgamated free product examples.
- *Chifan, Diaz-Arias, Drimbe 2020, 2021:* More AFP, direct product, wreath product examples.
- *Chfian, Ioana, Osin, Sun 2021:* First property (T) examples.
- *D., Vaes 2024:* Twisted groups and virtual isomorphisms.
- *D., Vaes 2024:* First examples with infinite center.
- *Chifan, Fernández Quero, Osin, Tan 2025:* First property (T) examples with infinite center.

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We call (A, Δ) a **compact quantum group** if A is a von Neumann algebra, $\Delta : A \to A \overline{\otimes} A$ is a faithful normal unital $*$-homomorphism that is co-associative (called the **comultiplication**):

$$(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta \ ,$$

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and there exists a faithful normal state φ on A that is left and right invariant:

$$(\mathrm{Id} \otimes \varphi)\Delta(a) = \varphi(a)1 = (\varphi \otimes \mathrm{Id})\Delta(a) \quad \text{for all } a \in A\,.$$

The state φ is called the **Haar state**. If it is tracial, (A, Δ) is of **Kac type**.

Examples of Kac type compact quantum groups

Two examples to keep in mind:

1. $L^\infty(K)$ for a compact group K, with

$$\Delta_K : L^\infty(K)\overline{\otimes}L^\infty(K) : (\Delta_K F)(g,h) = F(gh)\,,$$

and Haar state given by integration w.r.t the Haar measure.

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2. $L(G)$ for a countable group G. Here we define

$$\Delta_G : L(G) \to L(G)\overline{\otimes}L(G) : u_g \mapsto u_g \otimes u_g.$$

   The Haar state is given by the canonical trace: $\tau(x) = \langle x\delta_e, \delta_e\rangle$.

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   Note that $(L(G), \Delta_G)$ is **co-commutative**: $\mathrm{Flip} \circ \Delta_G = \Delta_G$.

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Note that for countable groups G and H: $(L(G), \Delta_G) \cong (L(H), \Delta_H)$ iff $G \cong H$.

Reason: $x \in L(G)$ satisfies $\Delta_G(x) = x \otimes x$ if and only if $x = u_g$ for some $g \in G$.

Quantum W*-superrigidity II

Definition

We say that a compact quantum group (A, Δ_A) is quantum W*-superrigid if the following holds: if (B, Δ_B) is any compact quantum group such that $B \cong A$ as von Neumann algebras, then $(A, \Delta_A) \cong (B, \Delta_B)$ as compact quantum groups.

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Converse is **not true**: none of the W*-superrigid wreath product groups of [IPV10, BV12, Ber14, DV24] are quantum W*-superrigid!

Cohomological obstructions

Let (A, Δ) be a Kac type compact quantum group. A **unitary 2-cocycle** is a unitary $\Omega \in A \overline{\otimes} A$ satisfying

$$(\Omega \otimes 1)(\Delta \otimes \mathrm{Id})(\Omega) = (1 \otimes \Omega)(\mathrm{Id} \otimes \Delta)(\Omega)\,.$$

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If Ω is a 2-cocycle, then

$$\Delta_\Omega : A \to A\overline{\otimes}A : \Delta_\Omega(a) = \Omega\Delta(a)\Omega^*$$

is again co-associative and (A, Δ_Ω) is again a Kac type compact quantum group.

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$\rightarrow$ Potentially **two different quantum group structures** on the same von Neumann algebra A.

No-go result

Proposition (D., Vaes '25)

Let G be an icc group, $G_0 < G$ a subgroup and $\Omega_0 \in L(G_0)\overline{\otimes}L(G_0)$ a nontrivial unitary 2-cocycle for $(L(G_0), \Delta_0)$. View Ω_0 as a unitary 2-cocycle Ω for $(L(G), \Delta)$. Then, Δ_Ω is not symmetric.
In particular, Ω_0 remains nontrivial as a 2-cocycle on $(L(G), \Delta)$, we have that $(L(G), \Delta_\Omega) \ncong (L(G), \Delta)$ and $(L(G), \Delta_G)$ is not quantum W-superrigid.*

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In particular, Ω_0 remains nontrivial as a 2-cocycle on $(L(G), \Delta)$, we have that $(L(G), \Delta_\Omega) \not\cong (L(G), \Delta)$ and $(L(G), \Delta_G)$ is not quantum W-superrigid.*

Corollary (D., Vaes '25)

Let G be an icc group with a finite abelian subgroup $G_0 < G$ such that $H^2(\widehat{G}_0, \mathbb{T}) \neq 0$. Then G is not quantum W-superrigid.*

Wreath product groups fail

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<u>Remark:</u> Not an issue for plain W*-superrigidity because of a vanishing result for **symmetric** 2-cocycles (IPV10).

$\rightarrow$ We need a general vanishing of cohomology result.

Co-induced left-right Bernoulli crossed product I

Main construction:

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- Consider the co-induced left-right Bernoulli action $\alpha : \Gamma \times \Gamma \curvearrowright (A, \tau) = (A_0, \tau_0)^\Gamma$ defined as follows: denote by $\pi_k : A_0 \to A$ the embedding in the k'th tensor factor. Then

$$\alpha_{(g,h)}(\pi_k(a)) = \pi_{gkh^{-1}}(\beta_g(a)) \, .$$

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- Define $M = (A, \tau) \rtimes_\alpha (\Gamma \times \Gamma)$.

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- Γ is in 'class $\mathcal{C}$': Γ is nonamenable, weakly amenable, bi-exact, and $C_\Gamma(g)$ is amenable for every $g \in \Gamma \setminus \{e\}$.
  Think: $\Gamma = \mathbb{F}_n$ for $2 \leq n \leq \infty$.

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Note: using Δ_0 and $\Delta_{\Gamma\times\Gamma}$, we can build Δ so that (M, Δ) is again naturally a Kac type compact quantum group.

Goal: find $\beta : \Gamma \curvearrowright (A_0, \Delta_0)$ so that (M, Δ) is quantum W*-superrigid.

Main result

Theorem (D., Vaes '25)

For each of the following action $\beta : \Gamma \curvearrowright (A_0, \Delta_0)$*, the co-induced left-right Bernoulli crossed product* (M, Δ) *is quantum W*-superrigid.*

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1. *Let* Λ *be a torsion free amenable icc group. Set* $(A_0, \Delta_0) = (L(\Lambda), \Delta_\Lambda)$*. Let* $\Gamma = \Lambda * \mathbb{Z}$ *act by* $\beta_g = \mathrm{Ad}\, u_g$ *for* $g \in \Lambda$ *and* $\beta_a = \mathrm{Id}$ *for* $a \in \mathbb{Z}$*.*

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2. *Let $(A_0, \Delta_0) = (L^\infty(K), \Delta_K)$ where $K = \mathbb{T}^n$ for $n \geq 3$ or for many finite groups K. Define $\Gamma = \mathbb{F}_{\mathrm{Aut}\, K}$ and set $\beta_\alpha(a) = \alpha(a)$ for every $\alpha \in \mathrm{Aut}\, K$.*

E.g. $K = \mathrm{SL}_2(\mathbb{F}_p)$ for p a prime and $p \geq 5$ or $K = \mathrm{SL}_n(\mathbb{F}_q)$ for q a prime power (three exceptions).

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2. Relative rigidity for compact quantum groups.
   Note: construction of M is functorial in $\Gamma \curvearrowright^{\beta} (A_0, \Delta_0)$.
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   $\rightarrow$ We should recover the quantum group structure from the von Neumann algebra A_0 and the quantum group automorphisms β_g.
3. Classification of coarse embeddings (D. Vaes '24).
   $\rightarrow$ Generalise comultiplication approach to quantum group setting.

Cohomology of crossed products 1

De Commer '10: Every unitary 2-cocycle of $(L(G), \Delta_G)$ is a coboundary if G is torsion free.

We extend this to the setting of a countable group acting on a Kac type compact quantum group $\Gamma \curvearrowright (A, \Delta_A)$ by quantum group automorphisms. Then the crossed product $M := A \rtimes \Gamma$ naturally becomes a Kac type compact quantum group by

$$\Delta_M(au_g) := \Delta_A(a)(u_g \otimes u_g)$$

and

$$\varphi_M(au_g) = 0, \qquad \forall g \in \Gamma \setminus \{e\}\,.$$

Cohomology of crossed products 2

Proposition

Let $M = A \rtimes_\alpha \Gamma$ as before.

1. *For every unitary 2-cocycle Ω on M, there is finite subgroup $\Lambda < \Gamma$ s.t. $\Omega \sim \Omega_0$ for a 2-cocycle $\Omega_0 \in (A \rtimes \Lambda)\overline{\otimes}(A \rtimes \Lambda)$.*
2. *If Ω_1, Ω_2 are unitary 2-cocycles for (A, Δ_A), then $\Omega_1 \sim \Omega_2$ as 2-cocycles for (M, Δ_M) iff there is a $g \in \Gamma$ s.t. $\Omega_1 \sim (\alpha_g \otimes \alpha_g)(\Omega_2)$ as 2-cocycles for (A, Δ_A).*
3. *Nontrivial 2-cocycles for (A, Δ_A) and $(L(\Gamma), \Delta)$ remain nontrivial as 2-cocycles for (M, Δ_M).*

Cohomology of tensor products

Let (A_k, Δ_k) be Kac type compact groups with haar states φ_k and set

$$(A, \varphi) := \overline{\bigotimes}_k (A_k, \varphi_k)\,.$$

Then A is again a Kac type compact quantum group with Δ defined by $\Delta \circ \pi_k = (\pi_k \otimes \pi_k) \circ \Delta_k$ for all k.

Proposition

Every unitary 2-cocycle for (A, Δ) is a coboundary iff the following hold:

1. *For every k, every unitary 2-cocycle on (A_k, Δ_k) is a coboundary.*
2. *if k, l are distinct, then any bicharacter $Z \in \mathcal{U}(A_k \overline{\otimes} A_l)$ is trivial.*

Vanishing of cohomology

Now let $M = (A_0, \tau_0)^\Gamma \rtimes_\alpha (\Gamma \times \Gamma)$ be our co-induced Bernoulli crossed products. Conclusion: M has no nontrivial unitary 2-cocycles when

- Γ is torsion free
- (A_0, Δ_0) has no nontrivial 2-cocycles or bicharacters.

Examples are

- $(L(\Lambda), \Delta_\Lambda)$ with Λ torsion free.
- $(L^\infty(K), \Delta_K)$ with $K = \mathbb{T}^n$, or connected compact abelian second countable.
- $(L^\infty(K), \Delta_K)$ with K finite, perfect, with trivial Shur multiplier, e.g. $K = \mathrm{SL}_n(\mathbb{F}_q)$ for many n and prime powers q.

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$\rightarrow$ We need additional rigidity.

We should recover Δ_0 from A_0 and the fact that all β_g are quantum group automorphisms.
$\rightarrow$ **Rigidity** of A_0 **relative to** a group of automorphisms $\Gamma < \mathrm{Aut}(A_0, \Delta_0)$.

Relative rigidity 2

Definition

We say (A, Δ_A) is rigid relative to $\Gamma \curvearrowright^{\alpha} (A, \Delta_A)$ if the following holds: if $\Gamma \curvearrowright^{\beta} (B, \Delta_B)$ is another action by quantum group automorphisms and $\pi : A \to B$ is a state preserving von Neumann algebra isomorphism s.t. $\beta_G \circ \pi = \pi \circ \alpha_g$ for all $g \in \Gamma$,

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Relative rigidity 2

Definition

We say (A, Δ_A) is rigid relative to $\Gamma \curvearrowright^{\alpha} (A, \Delta_A)$ if the following holds: if $\Gamma \curvearrowright^{\beta} (B, \Delta_B)$ is another action by quantum group automorphisms and $\pi : A \to B$ is a state preserving von Neumann algebra isomorphism s.t. $\beta_G \circ \pi = \pi \circ \alpha_g$ for all $g \in \Gamma$, then there exists a quantum group isomorphisms $\pi_0 : (A, \Delta_A) \to (B, \Delta_B)$ and a group automorphism ζ of Γ s.t. $\beta_{\zeta(g)} \circ \pi_0 = \pi_0 \circ \alpha_g$ for all $g \in \Gamma$.

A finite group K is rigid relative to $\mathrm{Aut}\, K$ if the following holds: whenever Λ is a group and $\pi : K \to \Lambda$ is a bijection s.t. $\pi \circ \alpha \circ \pi^{-1} \in \mathrm{Aut}\, \Lambda$ for all $\alpha \in \mathrm{Aut}\, K$, then $K \cong \Lambda$.

Examples of relative rigidity

Some examples of relative rigidity are:

1. K^n relative to $\mathrm{SL}_n(\mathbb{Z})$, for any second countable connected compact abelian K and $n \geq 3$.

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Non-example: a **non-abelian** compact second countable group K is **never** rigid relative to any $\Gamma < \mathrm{Aut}(K)$.

Comultiplication approach 1

Let $M = (A_0, \tau_0)^\Gamma \rtimes_\alpha (\Gamma \times \Gamma)$ be our co-induced left-right Bernoulli crossed product and let (Q, Δ_Q) be another compact quantum group. Suppose $\pi : M \to Q$ is a normal $*$-isomorphism.

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Consider $\Delta_1 := (\pi^{-1} \otimes \pi^{-1}) \circ \Delta_Q \circ \pi : M \to M \overline{\otimes} M$. Tautologically, (M, Δ_1) and (Q, Δ_Q) are isomorphic as quantum groups.

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Step 1: Analyse Δ_1

→ Classification of coarse embeddings

→ Using Popa's deformation/rigidity framework

Comultiplication approach 2

Result: we find

- unitary $\Omega \in M \overline{\otimes} M$,
- a character $\chi : \Gamma \times \Gamma \to \mathbb{C}$,
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- $\Omega\Delta_1(\pi_e(A_0))\Omega^* \subset \pi_e(A_0)\overline{\otimes}\pi_e(A_0)$,
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We get a new compact quantum group (M, Φ) with $\Phi = \Omega \Delta_1 \Omega^*$.

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We now use our vanishing of cohomology results to conclude that Ω is a coboundary. Hence (M, Φ) and (M, Δ_1) are isomorphic. We conclude

$$(M, \Delta) \cong (M, \Phi) \cong (M, \Delta_1) \cong (Q, \Delta_Q) .$$